Unruh Effect
Mukhanov and Winitzki Chap 8 Notes

Robert D. Klauber December 2, 2016
www.quantumfieldtheory.info

Refs:
Introduction to Quantum Effects in Gravity, Mukhanov, V., and Winitzki, S. (Cambridge, 2007)
Student Friendly Quantum Field Theory, Klauber, R.D., (Sandtrove 2015, 2nd ed, 3rd printing)

NOTE: All Section numbers, all section headings, and equation numbers of form (8.X) are with reference to Mukhanov and Winitzki (M&W), Chap. 8.

TYPOS:

pg. 241. top two lines of equation. signs before v_r everywhere should be reversed. (But not before v_r^2 .)

pg. 102, paragraph above (8.25). I believe "Condition (8.20)" reads better as "Condition (8.20) in (8.21)".

pg. 103. Fig. 8.2 text. I believe the next to last sentence saying the lightcone in the *t*-*x* coordinates (dashed lines) has $\xi^1 = -a^{-1}$ is wrong and instead of that expression, it should be $\xi^0 = \pm\infty$. See my notes below for this figure.

pg. 107, (8.40). I believe after the first equal sign one needs to insert a factor $\pm\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}$.

pg. 108, (8.47). There should be a *k* (Boltzmann constant) in the denominator of the RHS

Wholeness Chart for this chapter

Note there is a wholeness chart summary of this entire chapter at the end of these notes. It should help to follow that along as you study the chapter.

Background for Chap 8

Other discoverers of the "Unruh" effect"

The Unruh effect was first described by Stephen Fulling in 1973, Paul Davies in 1975 and W. G. Unruh in 1976, and is sometimes called the Fulling–Davies–Unruh effect. It appears that Unruh deduced the Unruh temperature (our final result on pg. 13) first, however. The Unruh temperature is considered to be the temperature of the vacuum that an accelerated observer would measure.

The Unruh temperature has the same form as the Hawking temperature for black holes (see M&W Chap. 9), which was discovered independently by Stephen Hawking about the same time, so is occasionally called the Hawking-Unruh temperature.

What is meant by "constant" acceleration

Note that the accelerating observer in this case has constant acceleration as measured by her, NOT as measured in the inertial frame of *t*-*x*. That is, the accelerating observer carries a spring-mass system that has the same displacement under the acceleration for all time. Although that means constant acceleration as seen by the accelerating observer (who measures her own velocity to be zero), in the inertial frame of *t*-*x* the accelerating observer is speeding up but can never go faster than *c*. So from the *t*-*x* frame, the speed of the accelerating observer approaches *c* ever more slowly. In other words, the acceleration does not seem constant from the point of view of the *t*-*x* frame.

Frames

There are two frames we need to distinguish between. Two dimensions of space are suppressed to simplify.

1. The inertial frame from which the accelerating object (called the accelerating observer here) is observed. The spacetime coordinates used are *t*-*x* ($x^\mu = x^0, x^1$).

2. The accelerating frame of the object (accelerating observer). Spacetime coordinates used are ξ^0 (time) and ξ^1 (space). M&W call this the co-moving frame. ($\xi^\mu = \xi^0, \xi^1$).

Coordinate systems in these frames

For each of these frames, two different coordinate systems are used. One is the usual spacetime coordinates as shown in 1 and 2 above. The other, whose primary purpose is ostensibly to simplify analysis, is called the lightcone coordinate system. We explain them graphically in Fig. 1 and in text further below. Listed, these are as follows.

A. Lightcone coordinates for the inertial frame of 1 above. Coordinates are u and v, which are mixtures of t and x, where the transformation to the new coordinates is defined by $u = t - x$ and $v = t + x$.

B. Lightcone coordinates for the co-moving (accelerating) frame of 2 above. Coordinates are $\tilde{u}$ and $\tilde{v}$, which are mixtures of ξ^0 and ξ^1, where the transformation to the new coordinates is defined by $\tilde{u} = \xi^0 - \xi^1$ and $\tilde{v} = \xi^0 + \xi^1$.

Explaining the lightcone coordinate system

Fig. 1 shows that the lightcone coordinate system is simply a rotation of each of the t and x axes by 90°. Any event with coordinates t and x in the usual spacetime coordinates has coordinates $u = t - x$ and $v = t + x$ in lightcone coordinates. The source of the name for these coordinates is simply that their axes correspond with the lightcone lines.

A similar relationship exists in the co-moving (accelerated) frame between the spacetime coordinates ξ^0 and ξ^1 and the lightcone coordinates $\tilde{u}$ and $\tilde{v}$.

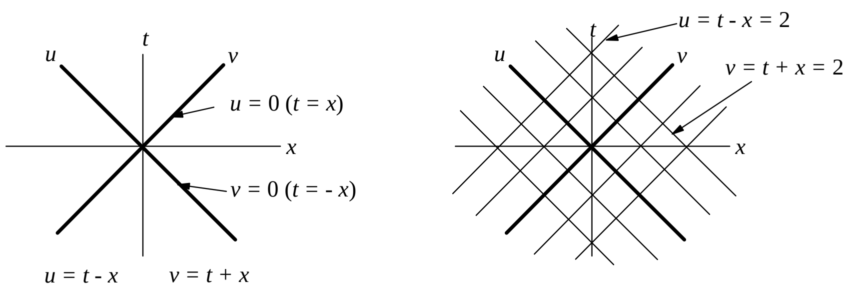

Figure 1. Usual *t-x* Coordinates vs Lightcone Coordinates *u-v*

Note that each value of constant u represents the world line a massless particle would have moving from left to right. Each line of constant v represents the world line for a massless particle moving from right to left. In this chapter, the authors only deal with quantized fields for massless particles. (See comment above (8.29) on pg. 103.)

Distinguish between 1) the "stationary" inertial observer frame t-x, 2) the accelerating observer's frame ξ^0-ξ^1, and 3) particles moving in spacetime as seen from each frame (which are massless in this chapter.)

A third kind of frame used in other texts

Note further that many texts employ a third inertial frame, actually a set of inertial frames, each having the same velocity as the accelerating object/observer at a particular instant. Each such frame is a different frame, as the object/accelerating observer has a different velocity at each instant. (There are an infinite number of these frames.) Each of these frames is typically called a co-moving frame, but that terminology is used in M&W for the frame of 2 above.

The assumption (the "locality" assumption of general relativity) is that the local inertial frame Minkowski coordinate system with the same velocity as an accelerating point can be used locally (at one 3D point at one instant, i.e, at one event) to accurately (on infinitesimal scales) represent the accelerating frame at that point. An instant later, we need another such Minkowski coordinate system to evaluate what is going on at that instant. Etc., etc. The results of all of these Minkowski coordinate systems can be merged to describe what is happening over time inside the accelerating frame. This locality assumption is assumed, under the surface, in the M&W treatment. The local inertial frame equivalency is not noted specifically.

Objectives of Sections 8.1 to 8.2

The whole purpose of Sects. 8.1 and 8.2 is to find the metric in the accelerating (co-moving) frame in terms of the spacetime coordinates ξ^0 and ξ^1. This is shown in the line element (8.28),

$$ds^2 = e^{2a\xi^1}\left(\left(d\xi^0\right)^2 - \left(d\xi^1\right)^2\right) \qquad \text{M\&W (8.28), pg. 102,} \qquad (1)$$

where the metric is

$$_{accel}\,g_{\mu\nu} = \begin{bmatrix} e^{2a\xi^1} & 0 \\ 0 & e^{2a\xi^1} \end{bmatrix} \xrightarrow{\text{in M\&W}} = \Omega^2\left(\xi^0,\xi^1\right) \text{ on pg. 100, } (8.15)\,. \qquad (2)$$

This is called Rindler spacetime.

Steps to Rindler spacetime metric

M&W do the following.

1. Define lightcone coordinates (Sect. 8.1, pg. 98)
2. Find the trajectory of a single point undergoing constant acceleration (as defined above) in the "stationary" observer coordinates *t*–*x*. (Sect 8.1 pg. 99).
3. Find trajectories in *t*–*x* space of every point in the accelerating frame (not just one point like in 2 above). ((8.27) of Sect. 8.2,pg. 102 and Fig. 8.2, pg. 103.)
4. Find the metric for the accelerating frame. ((8.15) of Sect 8.2, pg. 100 [our (2)] and (8.28), pg. 102 [our (1).

Objectives of Sections 8.3 to 8.5

The results of Sects. 8.2 are quantized in Sect 8.3 with concomitant results shown in Sects. 8.4 and 8.5

8.1 Accelerated motion

Lightcone coordinates (pg. 98)

Lightcone coordinates are supposed to simplify analysis, but I believe they make it harder for newcomers to understand what is going on. It is one more level of abstraction, for an already abstract phenomenon.

I would prefer teaching new students this material by following Misner, Thorne, and Wheeler's *Gravitation* as referenced in Appendix A. It doesn't use the lightcone analysis, which can be more confusing IMO. It is more straightforward.

However, for those who wish to go the M&W route (or are forced to by the way their class is being taught), I include some notes along the way below for the *u*-*v* lightcone coordinates.

From M&W (8.6)

$$u = t - x \qquad v = t + x\,, \qquad \text{M\&W (8.6), pg 96} \qquad (3)$$

$$du = dt - dx \qquad dv = dt + dx\,, \qquad (4)$$

and

$$\left.\begin{array}{ll} u = t - x & v = t + x \\ \underline{v = t + x} & \underline{-u = -t + x} \\ u + v = 2t & v - u = 2x \end{array}\right\} \qquad t = \frac{u+v}{2} \qquad x = \frac{-u+v}{2}\,, \qquad (5)$$

$$dt = \frac{du+dv}{2} \qquad dx = \frac{-du+dv}{2}\,. \qquad (6)$$

So with (6)

$$\begin{aligned} ds^2 = dt^2 - dx^2 &= \left(\frac{du+dv}{2}\right)^2 - \left(\frac{-du+dv}{2}\right)^2 \\ &= \left(\frac{du}{2}\right)^2 + \left(\frac{dv}{2}\right)^2 + 2\frac{du}{2}\frac{dv}{2} - \left(\frac{du}{2}\right)^2 - \left(\frac{du}{2}\right)^2 + 2\frac{du}{2}\frac{dv}{2} = du\,dv\,. \qquad \text{M\&W } (8.7) \text{ LHS} \end{aligned} \qquad (7)$$

which looks a little unusual in terms of what ds^2 looks like in coordinate systems we are more familiar with (as that of the first equal sign in the first row of (7)).

If on the RHS of (8.7) we take

$$dx^\alpha = \begin{bmatrix} du \\ dv \end{bmatrix} \qquad \to \qquad \begin{bmatrix} du & dv \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}}_{g^{(M)}_{\alpha\beta}} \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} du & dv \end{bmatrix} \begin{bmatrix} \frac{dv}{2} \\ \frac{du}{2} \end{bmatrix} = \frac{dudv}{2} + \frac{dvdu}{2} = dudv\,. \qquad (8)$$

So, from (7) and (8), we get (8.7) and (8.8).

Solution to Exercise 8.1, (Prob. on pg 99. Solution on pgs 240-41)

It is not mentioned in M&W, but one must assume the following relations for tilde inertial light cone coordinates, paralleling relations (8.6), pg. 98 for the non tilde inertial light cone coordinates, hold.

$$\tilde{u} = \tilde{t} - \tilde{x} \qquad \tilde{v} = \tilde{t} + \tilde{x} \qquad \left(\tilde{t} = \xi^0 \quad \tilde{x} = \xi^1 \ \text{ M\&W use tildes in solution and Greek letters in text} \right) \quad (9)$$

Then use the Lorentz transformation at the bottom of pg. 240 in (9)

$$\begin{aligned} \tilde{u} = \tilde{t} - \tilde{x} &= \frac{t - v_r x}{\sqrt{1-v_r^2}} - \frac{x - v_r t}{\sqrt{1-v_r^2}} = \frac{(1+v_r)t - (1+v_r)x}{\sqrt{1-v_r^2}} = \frac{(1+v_r)(t-x)}{\sqrt{1-v_r^2}} = \frac{(1+v_r)}{\sqrt{1-v_r^2}} u = \alpha u \\ \tilde{v} = \tilde{t} + \tilde{x} &= \frac{t - v_r x}{\sqrt{1-v_r^2}} + \frac{x - v_r t}{\sqrt{1-v_r^2}} = \frac{(1-v_r)t + (1-v_r)x}{\sqrt{1-v_r^2}} = \frac{(1-v_r)(t+x)}{\sqrt{1-v_r^2}} = \frac{(1-v_r)}{\sqrt{1-v_r^2}} v = \frac{1}{\alpha} v \end{aligned} \qquad (10)$$

$$\alpha = \frac{1+v_r}{\sqrt{1-v_r^2}} = \frac{1+v_r}{\sqrt{(1+v_r)(1-v_r)}} = \frac{\sqrt{1+v_r}}{\sqrt{1-v_r}} \qquad (11)$$

(10) and (11) disagree with M&W by the sign in front of the v_r.

M&W analysis:

$$\alpha^2 = \frac{1-v_r}{1+v_r} \ \to \ \alpha^2 (1+v_r) = 1 - v_r \ \to \ -\alpha^2 - \alpha^2 v_r + 1 = v_r \ \to \ -\alpha^2 + 1 = v_r + \alpha^2 v_r \ \to \ v_r = \frac{1-\alpha^2}{1+\alpha^2} \quad (12)$$

present analysis:

$$\alpha^2 = \frac{1+v_r}{1-v_r} \ \to \ \alpha^2 (1-v_r) = 1 + v_r \ \to \ \alpha^2 - \alpha^2 v_r - 1 = v_r \ \to \ v_r + \alpha^2 v_r = -1 + \alpha^2 \ \to \ v_r = -\frac{1-\alpha^2}{1+\alpha^2} \quad (13)$$

The sign on v_r in the Lorentz transformations at the bottom of page 240 can be + or – depending on the direction (minus x or plus x) of the second frame relative to the first. (The Lorentz factor $\sqrt{1-v_r^2}$ is always the same since the sign of v_r squared is always positive. So, either way is OK, but the transformations at bottom of pg 240 should use + sign in front of v_r to be consistent with the relations as printed at the top of pg. 241.

8.2 Comoving frame of accelerated observer

Getting (8.28), pg. 102

The line element metric in the original inertial Minkowski frame is

$$ds^2 = dt^2 - dx^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \qquad \text{M\&W (8.1), pg. 97} \qquad (14)$$

$$t = \frac{1}{a} e^{a\xi^1} \sinh a\xi^0 \qquad x = \frac{1}{a} e^{a\xi^1} \cosh a\xi^0 \qquad \text{M\&W (8.27), pg. 102} \qquad (15)$$

From (8.27) [our (15)],

$$dt = e^{a\xi^1}\, sinh\, a\xi^0 d\xi^1 + e^{a\xi^1}\, cosh\, a\xi^0 d\xi^0 \qquad dx = e^{a\xi^1}\, cosh\, a\xi^0 d\xi^1 + e^{a\xi^1}\, sinh\, a\xi^0 d\xi^0 \tag{16}$$

$$\begin{aligned} dt^2 &= e^{2a\xi^1}\, sinh^2\, a\xi^0 \left(d\xi^1\right)^2 + e^{2a\xi^1}\, sinh\, a\xi^0\, cosh\, a\xi^0 d\xi^0 d\xi^1 + e^{2a\xi^1}\, cosh^2\, a\xi^0 \left(d\xi^0\right)^2 \\ dx^2 &= e^{2a\xi^1}\, cosh^2\, a\xi^0 \left(d\xi^1\right)^2 + e^{2a\xi^1}\, sinh\, a\xi^0\, cosh\, a\xi^0 d\xi^0 d\xi^1 + e^{2a\xi^1}\, sinh^2\, a\xi^0 \left(d\xi^0\right)^2 \end{aligned} \tag{17}$$

$$\begin{aligned} dt^2 - dx^2 &= e^{2a\xi^1}\, sinh^2\, a\xi^0 \left(d\xi^1\right)^2 + e^{2a\xi^1}\, cosh^2\, a\xi^0 \left(d\xi^0\right)^2 - e^{2a\xi^1}\, cosh^2\, a\xi^0 \left(d\xi^1\right)^2 + e^{2a\xi^1}\, sinh^2\, a\xi^0 \left(d\xi^0\right)^2 \\ &= e^{2a\xi^1} \underbrace{\left(sinh^2\, a\xi^0 - cosh^2\, a\xi^0\right)}_{=-1} \left(d\xi^1\right)^2 + e^{2a\xi^1} \underbrace{\left(cosh^2\, a\xi^0 - sinh^2\, a\xi^0\right)}_{=1} \left(d\xi^0\right)^2 \\ &= e^{2a\xi^1}\left(\left(d\xi^0\right)^2 - \left(d\xi^1\right)^2\right) \qquad \text{M\&W } (8.28)\text{, pg. 102.} \end{aligned} \tag{18}$$

Fig. 8.2, pg. 103

In the text under the figure it says “The dashed lines show the lightcone which corresponds to $\xi^1 = -a^{-1}$”. However consider (8.27) [our (15)]. Since on the light cone, $x = \pm\, t$, we have

$$t = \pm x \quad \rightarrow \frac{1}{a} e^{a\xi^1}\, sinh\, a\xi^0 = \frac{1}{a} e^{a\xi^1}\, cosh\, a\xi^0 \;\rightarrow\; \frac{e^{a\xi^0} - e^{-a\xi^0}}{2} = \frac{e^{a\xi^0} + e^{-a\xi^0}}{2} \;\rightarrow\; \xi^0 = \pm\infty, \tag{19}$$

for the lightcone (dashed) lines in Fig. 8.2. This makes sense from the figure, since the x axis is for $\xi^0 = 0$ and the dotted lines increase the magnitude of ξ^0 as one moves away from the x axis. The limits on ξ^0 are $+\infty$ and $-\infty$ (see (relation on pg. 102 2/3 down the page), and the dashed lines represent the limiting case for ξ^0.

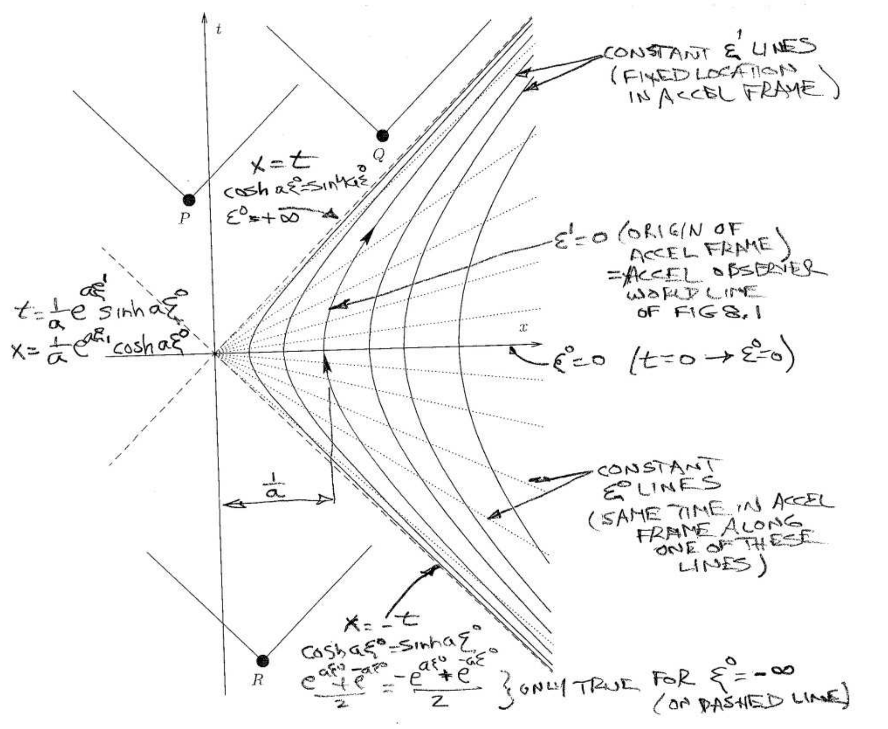

Figure 2. M&W Figure 8.2 with Comments

8.3 Quantum fields in inertial and accelerated frames

$$S[\phi]=\frac{1}{2}\int g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{-g}\;d^2x \qquad \text{M\&W (8.29), pg. 103} \qquad (20)$$

Statement at top of page 104

Note the value of S in (8.29) [our (20)] is invariant under any transformation, as S is a world scalar. However, the form of the integral (not just its value) will be invariant (look the same in the new coordinates as it did in the old) under some transformations, but not others.

To say S is "conformally invariant" here means the form of the integral is invariant under a conformal transformation. The transformation from the "stationary" inertial frame to the accelerating frame, as in (14) to (18), is conformal because at any event, the coordinate axes differences dt and dx stretch by the same ratio $e^{a\xi^1}$ to $e^{a\xi^1}dt$ and $e^{a\xi^1}dx$. So, any angle between any two crossing lines at that event will remain unchanged under the transformation. (See the link "Conformal and Scale Invariant Transformations" at the website under the title on pg. 1.)

Here the new system is the accelerated system using ξ^0 and ξ^1 spacetime coordinates. The old system is the inertial system t and x spacetime coordinates.

Metric change under transformation from inertial to accelerated frame

2D case (1+1 D)

In the t-x coordinates of the inertial frame of (8.1)

$$ds^2=dt^2-dx^2=\eta_{\alpha\beta}dx^\alpha dx^\beta \qquad g_{\alpha\beta}=\eta_{\alpha\beta}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad \text{M\&W (8.1), pg. 97.} \qquad (21)$$

In the ξ^0 and ξ^1 coordinates of the accelerated frame of (8.28) [our (18)],

$$dt^2-dx^2=e^{2a\xi^1}\left(\left(d\xi^0\right)^2-\left(d\xi^1\right)^2\right) \qquad \tilde{g}_{\alpha\beta}=\begin{bmatrix}e^{2a\xi^1} & 0\\ 0 & -e^{2a\xi^1}\end{bmatrix}=e^{2a\xi^1}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=e^{2a\xi^1}\eta_{\alpha\beta}=\underbrace{e^{2a\xi^1}}_{\Omega^2}g_{\alpha\beta} \qquad (22)$$

So,

$$\sqrt{-g}=1 \qquad \sqrt{-\tilde{g}}=\sqrt{\left(e^{2a\xi^1}\right)^2}=e^{2a\xi^1}=\Omega^2=\Omega^2\sqrt{-g} \qquad (23)$$

The matrix representing $\tilde{g}^{\alpha\beta}$ is just the inverse of $\tilde{g}_{\alpha\beta}$, hence

$$\tilde{g}^{\alpha\beta}=\Omega^{-2}g^{\alpha\beta} \qquad \text{M\&W (8.30) pg. 104.} \qquad (24)$$

Thus, in the integrand of (8.29) [our (20)], we have

$$\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{-\tilde{g}}=\Omega^{-2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\Omega^2\sqrt{-g}=g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{-g}\,, \qquad (25)$$

which has the same form in both systems at any 2D point in the system. Thus (8.29) [our (20)] is invariant in form under the transformation. This gives us (8.31).

4D case (3+1)

Note that in full 4D spacetime the metric (for acceleration only in ξ^1 direction) is

$$dt^2-dx^2-dy^2-dz^2=e^{2a\xi^1}\left(\left(d\xi^0\right)^2-\left(d\xi^1\right)^2\right)-d\xi^2-d\xi^3 \qquad {}_{4\mathrm{D}}\tilde{g}_{\alpha\beta}=\begin{bmatrix}e^{2a\xi^1} & & & \\ & -e^{2a\xi^1} & & \\ & & -1 & \\ & & & -1\end{bmatrix}\neq\underbrace{e^{2a\xi^1}}_{\Omega^2}\,{}_{4\mathrm{D}}g_{\alpha\beta}\,. \qquad (26)$$

But we still have, as in the 2D case,

$$\sqrt{-\tilde{g}_{4\text{D}}} = \sqrt{\left(e^{2a\xi^1}\right)^2} = e^{2a\xi^1} = \Omega^2 = \Omega^2\sqrt{-g_{4\text{D}}}\,, \tag{27}$$

so we don't get the cancellation of factors, as in (25).

Conclusion:

The 2D case is conformally invariant, but the 4D case is not.

Getting from M&W (8.31) to the equation following it

$$S = \tfrac{1}{2}\int \left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right)dtdx \qquad \text{M\&W (8.31)} \tag{28}$$

We only need to find the integrand in terms of *u* and *v* derivatives as that gives us the value at a point in spacetime (which has *t* and *x* values in one system and *u* and *v* values in another). We then integrate over the infinitesimal volume (2D really in our case of suppressing two dimensions) *dudv* in the *u*-*v* coordinate system over all spacetime (again "all" means in one spatial dimension plus time).

Note

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial t} \qquad \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}\,. \tag{29}$$

From M&W(8.6) [our (3)],

$$\frac{\partial u}{\partial t} = 1 \qquad \frac{\partial v}{\partial t} = 1 \qquad \frac{\partial u}{\partial x} = -1 \qquad \frac{\partial v}{\partial x} = 1\,. \tag{30}$$

Using (30) in (29), we have

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v} \qquad \frac{\partial\phi}{\partial x} = -\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}\,. \tag{31}$$

(31) in the integrand of (8.31) [our (28)] yields

$$\begin{aligned}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 &= \left(\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}\right)^2 - \left(-\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}\right)^2 \\ &= \left(\frac{\partial\phi}{\partial u}\right)^2 + \left(\frac{\partial\phi}{\partial v}\right)^2 + 2\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} - \left(\frac{\partial\phi}{\partial u}\right)^2 - \left(\frac{\partial\phi}{\partial v}\right)^2 + 2\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} = 4\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v}.\end{aligned} \tag{32}$$

Using (32) as our value of the integrand in (28), we then integrate this over *u*-*v* coordinates to get *S* of (8.31) [our (28)].

$$S = \tfrac{1}{2}\int 4\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v}dudv = 2\int \frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v}dudv \qquad \text{M\&W unlabeled equation mid pg 104 LHS.} \tag{33}$$

To get the RHS of the 2nd row of the unlabeled equation mid pg 104, we simply follow the above steps for the bottom row of M&W (8.31) for the accelerated system (which has ξ^0 and ξ^1 spacetime coordinates and $\tilde{u}$ and $\tilde{v}$ lightcone coordinates).

Getting solution form for ϕ at bottom of pg. 104

First note that the Ω in the last equation on pg 104 is different from the Ω of equation (8.30) [our (24)].

Massless fields/particles

Note also that because we are dealing with massless particles, from

$$m^2 = \omega^2 - \mathbf{p}^2 = \omega^2 - k^2 = 0 \qquad \rightarrow \qquad k = \pm\omega\,, \tag{34}$$

the unlabeled equations at the bottom of pg 104 are seen to be (we will discuss finding these relations below),

$$\phi \propto e^{-i\omega(t-x)} = e^{-i(\omega t - kx)} = e^{-i\omega u} \quad (\text{left moving}) \qquad e^{-i\omega(t+x)} = e^{-i(\omega t + kx)} = e^{-i\omega v} \quad (\text{right moving})$$

$$\phi \propto e^{-i\Omega(\xi^0-\xi^1)} = e^{-i(\Omega\xi^0 - K\xi^1)} = e^{-i\omega\tilde{u}} \quad (\text{left moving}) \qquad e^{-i\Omega(t+x)} = e^{-i(\Omega t + Kx)} = e^{-i\omega\tilde{v}} \quad (\text{right moving}). \tag{35}$$

In (35), the particles represented by the right moving solutions have negative values for 3-momentum, so they move in the opposite direction of the left moving particles. (See (8.33) where integration is only over positive k (i.e., ω) values, so the negative direction particles are picked up by the right moving solutions.

The easy way to find ϕ

The solution forms (35) are obtained by M&W using the lightcone coordinates systems u-v and $\tilde{u}-\tilde{v}$. I believe it is far easier for students to see how these solutions arises from the spacetime coordinate systems t-x and ξ^0 and ξ^1. To do that, start with the Lagrangian (the integrand) of (8.31) [our (28)],

$$\mathcal{L} = \tfrac{1}{2}\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right), \tag{36}$$

and plug it into the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \rightarrow \quad \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\phi_{,x}} = 0 \quad \rightarrow \quad \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \quad \rightarrow \quad \phi \propto e^{\pm i(\omega t - kx)}. \tag{37}$$

A parallel development yields the same form for ϕ in the accelerating frame.

The more complicated way to find ϕ

Alternatively, which M&W do, one can use the u-v coordinate system and take the Lagrangian as the integrand of the relation in the middle of pg 104 [our (33)].

$$\mathcal{L} = 2\phi_{,u}\,\phi_{,v}\,, \tag{38}$$

and use that in the Euler-Lagrange equation

$$\frac{\partial}{\partial u}\frac{\partial\mathcal{L}}{\partial\phi_{,u}} + \frac{\partial}{\partial v}\frac{\partial\mathcal{L}}{\partial\phi_{,v}} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \rightarrow \quad \partial_u\partial_v\phi = 0 \quad \rightarrow \quad \phi \propto e^{\pm i\omega u} + e^{\pm i\omega v}. \tag{39}$$

A parallel development yields the same form for ϕ in the accelerating frame

Note on the lightcone coordinates approach

The lightcone approach will only work for massless particles, for which $\omega = k$, because in that case $\omega t - kx = \omega(t-x) = \omega u$ (with comparable results for v). This seems to be another reason (other than simplicity for new learners) to avoid the lightcone analysis approach.

Comment at top of pg 105

"The actions in (8.31) [our (28) have a canonical form." This essentially means we can use the standard canonical quantization approach, which entails interpretation of the $\hat{a}^{\pm}_{\omega}$ and $\hat{b}^{\pm}_{\omega}$ coefficients as operators with the commutation relations (8.34).

Comment on M&W (8.33)

Note the equal sign between the top and bottom rows of (8.33) is due to $\hat{\phi}$ being a world scalar. Classically, this means it has the same numerical value at a given event, as seen from any frame. Thus, it is the same in the Minkowski frame (top row) and the Rindler frame (bottom row). This equality is assumed to carry over when we quantize, i.e., when the classical field ϕ (with a numeric value at each point) turns into a quantum field $\hat{\phi}$ composed of operators.

The equivalence in *form* of the action/Lagrangian in (8.31) for both frames leads to the equivalence in form of the top and bottom rows of (8.33) (not just that they are equal in value, but that their form is the same).

Remark: Rindler vacuum (pgs 105-106)

Significance of the quantities in the top line of pg 106

The quantities, for the inertial and accelerating frames respectively,

$$\left(\partial_u\hat{\phi}\right)^2 \qquad \left(\partial_{\tilde{u}}\hat{\phi}\right)^2 \tag{40}$$

represent energy density, since if one uses (8.33) in (40), one finds (after a fair amount of algebra[1])

$$\int \left(\partial_u\hat{\phi}\right)^2 dudv = \int \omega\left(\hat{a}^{+}_{\omega}\hat{a}^{-}_{\omega} + \tfrac{1}{2}\delta(0)\right)d\omega = \int \omega\left(N_a(\omega) + \tfrac{1}{2}\delta(0)\right)d\omega = H_M\,, \tag{41}$$

where we note the vacuum half quanta term. A parallel relation exists for the accelerating frame. Using the spacetime coordinates approach instead of the lightcone coordinates approach we could have found the Hamiltonian density from the Lagrangian density, integrated that over all space, and found the RHS of (41), as well.

Deducing a relation we will need

From (8.25), pg. 102,

$$u = -\frac{1}{a}e^{-a\tilde{u}} \qquad\qquad \text{M\&W (8.25).} \tag{42}$$

So,

$$\frac{\partial u}{\partial \tilde{u}} = e^{-a\tilde{u}} = \underbrace{-au}_{\text{from (8.25)}} \qquad \rightarrow \qquad \frac{\partial \tilde{u}}{\partial u} = -\frac{1}{au}\,. \tag{43}$$

Relation (8.36)

We get the last part of (8.36) [our (44)] from (8.35), pg. 106, and (43).

$$\underbrace{\left\langle 0_R\left|\left(\partial_u\hat{\phi}\right)^2\right|0_R\right\rangle}_{\substack{\text{Energy density expectation}\\ \text{value } e_M \text{ of R vacuum}\\ \text{(measured in M frame)}}} = \left(\frac{\partial\tilde{u}}{\partial u}\right)^2 \underbrace{\left\langle 0_R\left|\left(\partial_{\tilde{u}}\hat{\phi}\right)^2\right|0_R\right\rangle}_{\substack{\text{Energy density expectation}\\ \text{value } e_M \text{ of R vacuum}\\ \text{(measured in R frame)}}} = \frac{1}{a^2u^2}\underbrace{\left\langle 0_M\left|\left(\partial_u\hat{\phi}\right)^2\right|0_M\right\rangle}_{\substack{\text{Energy density expectation}\\ \text{value } e_M \text{ of M vacuum}\\ \text{(measured in M frame)}}} \qquad \text{M\&W (8.36).} \tag{44}$$

The operator $\left(\partial_u\hat{\phi}\right)^2$ represents the detector in the M frame (detecting energy). The operator $\left(\partial_{\tilde{u}}\hat{\phi}\right)^2$ represents a similar detector in the R frame. Each such detector can measure the expectation value of the energy density for a state of *a* particles (M frame particles) or the energy density for a state of *b* particles (R frame particles). In the case of (8.36) [our (44)], the state being measured is the vacuum state, for which one assumes a non zero value.

The LHS of (8.36) [our (44)] represents a measurement (to be precise, the expectation value of a measurement) of the Rindler vacuum by a detector in the Minkowski frame. It equals the RHS, which comprises a factor $1/a^2u^2$ times the measurement of the Minkowski vacuum energy density by a Minkowski frame detector.

From the top part of Fig. 3, after $t = 0$, we can see that as the R frame goes faster and faster (relative to the M frame), u gets smaller. Hence, the RHS of (8.36) [our (44)] gets larger. This means the lowest energy state (the vacuum) of the R frame looks, from the M frame, to have more and more energy as time goes on (starting from $t = 0$). This should not be too surprising, since a system going faster and faster is gaining more and more energy as time goes on.

From (8.36) [our (44)], we see that as the R frame approaches the speed of light ($u = 0$, as seen in Fig. 3), the energy density of the R frame vacuum approaches infinity, which is essentially what M&W say in this section. The same is true as one goes back into the distant past. u then approaches zero. (The incoming R frame was close to the speed of light and is decelerating as time approaches $t = 0$.) At $t = 0$, $u = -2$ in Fig. 3, where $a = ½$ for the R frame observer, so the denominator of

[1] See Klauber, Sect. 3.4.1, pg. 53-54 for a parallel treatment in spacetime coordinates that can make this a bit more apparent. The treatment there is for charged particles. That is, there are *a* type particles and *b* type particles (antiparticles, actually). In M&W, particles are chargeless, and thus, are their own antiparticles. In M&W Chap 8, the *b* particle notation is used for particles in the Rindler frame, *not* for antiparticles, as in Klauber.

the factor on the RHS of (8.36) [our (44)] equals 1. At that time, the two frames are instantaneously at rest with respect to each other, so they share the same vacuum state, and energy density expectation values of the vacuum for each are equal.

Using (8.36) [our (44)], one should be able to understand the lower part of Fig. 3.

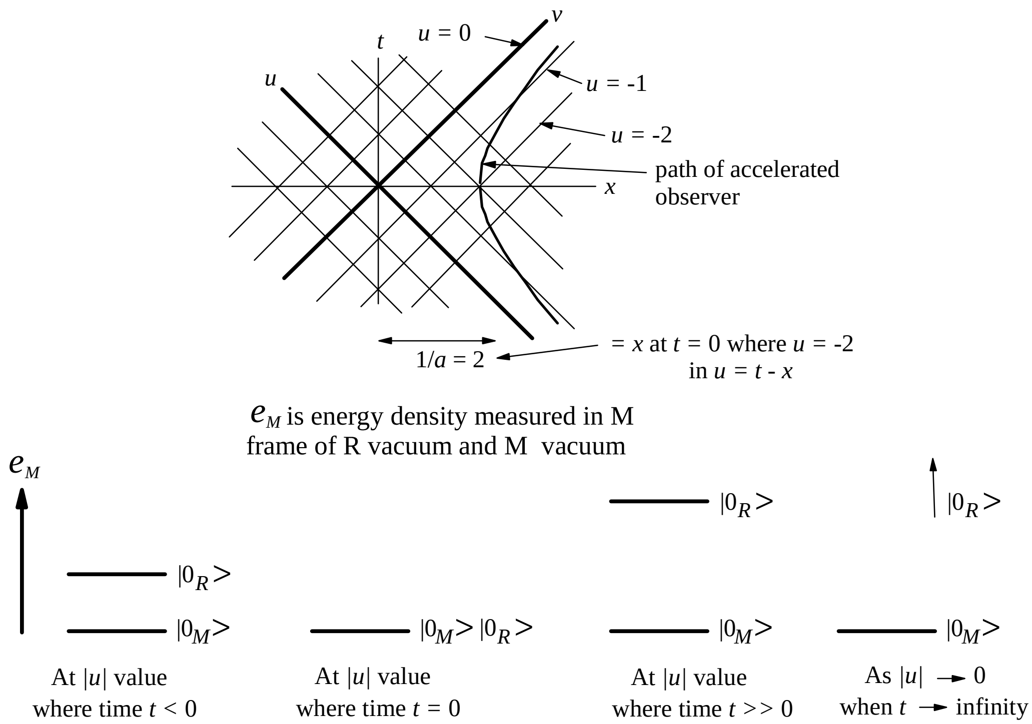

Figure 3. Showing M&W Equation (8.36) Graphically

In the limiting case where $a = 0$, the intersection of the accelerated observer (not really accelerating in this case) with the x axis occurs at $u = -\infty$, and the denominator in the RHS of (8.36) [our (44)] is equal to one. Thus, the vacuum of the R and M frames for that case is identical, as one would expect.

Reflections of the present author (not in M&W)

It seems one should expect the same effect we have seen above in reverse. That is, if we were to measure e_R, the energy density measured in the R frame of the R and M vacuums, we should get the same sort of relations in the bottom half of Fig. 3 with M and R trading places everywhere. When there is no relative velocity between the two frames, i.e., at $t = 0$, the *M* vacuum in the R frame would appear equal to the R vacuum in the R frame. At other times there would be relative velocity between the two, and at all times, the M frame would appear to be accelerating relative to the R frame. (This is a kinematic effect, it seems, and independent of whether one frame feels a force of acceleration or not.)

In other words, if we were to measure the M vacuum energy density in the R frame, we would see it as greater than that of the R vacuum measured in the R frame (except at $t = 0$, when they would be the same).

Additionally, as one can see from (8.36) [our (44)], if *M* vacuum energy really is zero, then the *R* vacuum energy would also be zero, regardless of the state of acceleration of the R frame. The above analysis appears to be related to the ½ quanta of the vacuum, which appear to be the only energy one would find in a pure vacuum.

Section 8.4 Bogolyubov transformations

Deriving (8.39)

From the lower row of (8.33) and using (8.37),

$$\begin{aligned}\hat{\phi} &= \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}}\frac{1}{\sqrt{2\Omega}}\left(e^{-i\Omega\tilde{u}}\hat{b}_\Omega^- + e^{i\Omega\tilde{u}}\hat{b}_\Omega^+\right) \qquad \text{lower row of M\&W (8.33)} \\ &= \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}}\frac{1}{\sqrt{2\Omega}}\left(e^{-i\Omega\tilde{u}}\underbrace{\int_0^\infty d\omega\left(\alpha_{\Omega\omega}\hat{a}_\omega^- - \beta_{\Omega\omega}\hat{a}_\omega^+\right)}_{\text{M\&W (8.37)}} + e^{i\Omega\tilde{u}}\underbrace{\int_0^\infty d\omega\left(\alpha_{\Omega\omega}^*\hat{a}_\omega^+ - \beta_{\Omega\omega}^*\hat{a}_\omega^-\right)}_{\text{complex conjug of M\&W (8.37)}}\right) \\ &= \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}}\int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}}\left(\left(e^{-i\Omega\tilde{u}}\alpha_{\Omega\omega}\hat{a}_\omega^- - e^{i\Omega\tilde{u}}\beta_{\Omega\omega}^*\hat{a}_\omega^-\right) - \left(e^{-i\Omega\tilde{u}}\beta_{\Omega\omega}\hat{a}_\omega^+ + e^{i\Omega\tilde{u}}\alpha_{\Omega\omega}^*\hat{a}_\omega^+\right)\right) \\ &= \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}}\int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}}\left(\left(e^{-i\Omega\tilde{u}}\alpha_{\Omega\omega} - e^{i\Omega\tilde{u}}\beta_{\Omega\omega}^*\right)\hat{a}_\omega^- - \left(e^{-i\Omega\tilde{u}}\beta_{\Omega\omega} + e^{i\Omega\tilde{u}}\alpha_{\Omega\omega}^*\right)\hat{a}_\omega^+\right).\end{aligned} \qquad (45)$$

Comparing the last row of (45) to the top row of (8.33) [our (46)]

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}}\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega u}\hat{a}_\omega^- + e^{i\omega u}\hat{a}_\omega^+\right) \quad \text{top row, (8.33),} \qquad (46)$$

gives (8.39).

$$\frac{1}{\sqrt{\omega}}e^{-i\omega\overbrace{(t-x)}^{u}} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}e^{-i\Omega'\overbrace{\left(\xi^0-\xi^1\right)}^{\tilde{u}}} - \beta^*{}_{\Omega'\omega}e^{i\Omega'\overbrace{\left(\xi^0-\xi^1\right)}^{\tilde{u}}}\right) \quad \text{M\&W (8.39)} \qquad (47)$$

<u>Deriving (8.40) top row first part</u>

From M&W (8.39) [our (47)] multiplied by $e^{i\Omega\tilde{u}}$,

$$\frac{1}{\sqrt{\omega}}e^{-i\omega u}e^{i\Omega\tilde{u}} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}e^{-i\Omega'\tilde{u}} - \beta^*{}_{\Omega'\omega}e^{i\Omega'\tilde{u}}\right)e^{i\Omega\tilde{u}} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}e^{-i(\Omega'-\Omega)\tilde{u}} - \beta^*{}_{\Omega'\omega}e^{i(\Omega'+\Omega)\tilde{u}}\right)$$

$$\begin{aligned}\int_{-\infty}^\infty d\tilde{u}\frac{1}{\sqrt{\omega}}e^{-i\omega u+i\Omega\tilde{u}} &= \int_{-\infty}^\infty d\tilde{u}\int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}e^{-i(\Omega'-\Omega)\tilde{u}} - \beta^*{}_{\Omega'\omega}e^{i(\Omega'-\Omega)\tilde{u}}\right) \\ &= \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}\int_{-\infty}^\infty d\tilde{u}\,e^{-i(\Omega'-\Omega)\tilde{u}} - \beta^*{}_{\Omega'\omega}\int_{-\infty}^\infty d\tilde{u}\,e^{i(\Omega'+\Omega)\tilde{u}}\right) \\ &= \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}2\pi\,\delta(\Omega'-\Omega) - \beta^*{}_{\Omega'\omega}2\pi\underbrace{\delta(\Omega'+\Omega)}_{\substack{\text{always}=0,\\ \text{as}\,\Omega>0,\,\Omega'>0}}\right) = \alpha_{\Omega'\omega}\frac{2\pi}{\sqrt{\Omega}}\ .\end{aligned} \qquad (48)$$

Thus, from the first part of the second line and the last part of (48),

$$\alpha_{\Omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^\infty d\tilde{u}\,e^{-i\omega u+i\Omega\tilde{u}} \qquad \text{M\&W (8.40) first part.} \qquad (49)$$

A parallel procedure leads to the $\beta_{\Omega\omega}$ of ((8.40), first part.

<u>Deriving (8.40) top row second part</u>

From (8.25),

$$u = -\frac{1}{a}e^{-a\tilde{u}} \quad \rightarrow \quad e^{-a\tilde{u}} = -au \qquad \text{M\&W (8.25) LHS,} \qquad (50)$$

$$du = e^{-a\tilde{u}}d\tilde{u} \quad \rightarrow \quad d\tilde{u} = \frac{1}{e^{-a\tilde{u}}}du = -\frac{1}{au}du\ . \qquad (51)$$

Also, from the RHS of (50)

$$e^{-a\tilde{u}} = -au \quad \rightarrow \quad \left(e^{-a\tilde{u}}\right)^{-\frac{i\Omega}{a}} = (-au)^{-\frac{i\Omega}{a}} \quad \rightarrow \quad e^{i\Omega\tilde{u}} = (-au)^{-\frac{i\Omega}{a}}\ . \qquad (52)$$

And,

$$\tilde{u} = -\infty \quad \rightarrow \quad u = -\infty \qquad\qquad \tilde{u} = +\infty \quad \rightarrow \quad u = 0 \tag{53}$$

Using (51) and (52) in (8.40) [our (49)], with integrations limits of (53), we have

$$\begin{aligned} \alpha_{\Omega'\omega} &= \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{-i\omega u} e^{i\Omega\tilde{u}} d\tilde{u} = \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} e^{-i\omega u}\left(-au\right)^{-\frac{i\Omega}{a}}\left(-\frac{1}{au}\right)du \\ &= \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} \left(-au\right)^{-\frac{i\Omega}{a}-1} e^{-i\omega u} du. \end{aligned} \tag{54}$$

A parallel procedure leads to the $\beta_{\Omega\omega}$ of ((8.40), second part.

Deriving (8.40) second row

We just use integral tables to get the 2nd row of (8.40).

Relation (8.42)

From the ± values of the first and last parts of (8.40), we get M&W's (8.42),

$$\left|\alpha_{\Omega\omega}\right|^2 = e^{\frac{2\pi\Omega}{a}}\left|\beta_{\Omega\omega}\right|^2 \qquad \text{M\&W (8.42).} \tag{55}$$

Relation (8.43)

In relation (8.43), pg. 107,

$$\begin{aligned} \left\langle \hat{N}_\Omega \right\rangle &= \left\langle 0_M \right| \hat{b}_\Omega^+ \hat{b}_\Omega^- \left| 0_M \right\rangle \qquad \left(\hat{b}_\Omega^+ \hat{b}_\Omega^- \text{ is R frame detector measuring } b \text{ particles in M vacuum}\right) \\ &= \left\langle 0_M \right| \underbrace{\left(\int d\omega\left(\alpha_{\omega\Omega}^* \hat{a}_\omega^+ - \beta_{\omega\Omega}^* \hat{a}_\omega^-\right)\right)\left(\int d\omega'\left(\alpha_{\omega'\Omega} \hat{a}_{\omega'}^- - \beta_{\omega'\Omega} \hat{a}_{\omega'}^+\right)\right)}_{\text{All terms yield zero except the one with } \hat{a}_\omega^- \hat{a}_{\omega'}^+ \left(\text{which} = \delta(\omega - \omega') + \hat{a}_{\omega'}^+ \hat{a}_\omega^-\right)} \left| 0_M \right\rangle \qquad \text{M\&W (8.43)} \\ &= \int d\omega \left|\beta_{\omega\Omega}\right|^2. \end{aligned} \tag{56}$$

we are measuring the number of b particles in the M vacuum as seen by a particle number detector in the R frame, which is sensitive to b particles. Note that we are ignoring the ½ quanta part of the number operator in (8.43) [our (56)].

Interpretation of (8.43)

As M&W note at the bottom of page 107, (8.43) [our (56)] is *interpreted* as the expectation value (mean number) of particles with frequency Ω found by the accelerated observer. I have no issues with any of the math in this chapter, but there may be questions about this interpretation, such as, 'what exactly does a detector taken from the M frame to the R frame measure?' And is all of the analysis applicable in reverse? That is, would an R frame detector brought into the M frame measure, according to the math, detect a number of particles in the vacuum with frequency ω? I am not a researcher in this field, have not delved into these questions, and would appreciate feedback from any reader with more insight into the issue (via the feedback link at the website shown at the beginning of these notes).

Deriving (8.45)

From (8.44) [our (57)]

$$\int_0^\infty d\omega\left(\left|\alpha_{\Omega\omega}\right|^2 - \left|\beta_{\Omega\omega}\right|^2\right) = \delta\left(0\right) \qquad \text{M\&W (8.44)} \tag{57}$$

and inserting (8.42) [our (55)], we have

$$\int_0^\infty d\omega\left(e^{\frac{2\pi\Omega}{a}}\left|\beta_{\Omega\omega}\right|^2 - \left|\beta_{\Omega\omega}\right|^2\right) = \int_0^\infty d\omega\left(e^{\frac{2\pi\Omega}{a}} - 1\right)\left|\beta_{\Omega\omega}\right|^2 = \delta\left(0\right), \text{ or} \tag{58}$$

$$\int_0^\infty d\omega \left|\beta_{\Omega\omega}\right|^2 = \frac{\delta(0)}{e^{\frac{2\pi\Omega}{a}} - 1} = \underbrace{\left\langle \hat{N}_\Omega \right\rangle}_{\substack{\text{from (8.43)} \\ \text{and LHS here}}} \qquad \text{M\&W (8.45).} \qquad (59)$$

Note on $\delta(0)$ and (8.46)

From

$$\delta^3(\mathbf{k}-\mathbf{k}') = \frac{1}{(2\pi)^3}\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{x} \quad \rightarrow \quad \delta^3(0) = \frac{1}{(2\pi)^3}\int e^0 d^3\mathbf{x} = \frac{V_\infty}{(2\pi)^3}, \qquad (60)$$

where V_∞ is the infinite volume of space, and M&W use δ, where we use δ^3. So when M&W equate $\delta(0)$ to V, they mean infinite volume and are ignoring a factor of $(1/2\pi)^3$ in (8.46) [our (61)]. Using (60) in (59),

$$\frac{\frac{V}{(2\pi)^3}}{e^{\frac{2\pi\Omega}{a}} - 1} = \left\langle \hat{N}_\Omega \right\rangle \quad \rightarrow \quad \left\langle n_\Omega \right\rangle = \frac{\left\langle \hat{N}_\Omega \right\rangle}{V} = \frac{1}{(2\pi)^3\left(e^{\frac{2\pi\Omega}{a}} - 1\right)} \qquad \text{M\&W (8.46) except for } (2\pi)^3 \text{ factor} \quad (61)$$

Bose-Einstein distribution

The probability density (for finding a particle at energy E in a collection of particles at temperature T) for bosons is the Bose-Einstein distribution, proportional to the RHS of (62),

$$\rho_{B-E} \propto \frac{1}{e^{\frac{E}{kT}} - 1}, \qquad (62)$$

where k is the Boltzmann constant.

Unruh temperature

Comparing (62) to (61), where $\Omega = E$,

$$T = \frac{a}{k2\pi} \ (\text{natural units}) \quad \rightarrow \quad T = \frac{a\hbar}{k2\pi c} \ (\text{MKS units}) \qquad \text{M\&W (8.47) except for typo } k \text{ factor} \quad (63)$$

It appears there is a typo in (8.47) where k is left out.

Note $\hbar$ = 6.63 X 10^{-34} J-s, k = 1.38 X 10^{-23} J/degree, and c = 3.00 X 10^{8} m/s. Meaning

$$T = \left(1.60\times10^{-17}\right)a \qquad (\text{MKS system}), \qquad (64)$$

and justifying M&W’s statement that the acceleration required to produce a measurable temperature is enormous. So we would need an enormous amount of energy to accelerate the frame of the accelerated observer to a level where a tiny Unruh temperature would be seen (representing a tiny amount of thermal energy in the particles causing that temperature.)

Summary of Chapter

As noted at the beginning, the wholeness chart at the end of these notes is a good summary of the entire chapter. An even briefer summary may be the following, which traces the chain of logic and analysis leading to the Unruh Effect.

Classical GR for accel frame $\rightarrow$ Rindler metric $\rightarrow$ ϕ has same value and form in R and M frames $\rightarrow$

quantize ($\hat{\phi}$ now an operator) $\rightarrow$ $\hat{b}_{\mathbf{k}}^{\pm}$ in terms of $\hat{a}_{\mathbf{k}}^{\pm}$ (Bogolyubov transf) $\rightarrow$ $\left|\alpha_{\Omega\omega}\right|^2 = e^{\frac{2\pi\Omega}{a}}\left|\beta_{\Omega\omega}\right|^2$ $\rightarrow$

normaliz cond, solve for $\int_0^\infty d\omega\left|\beta_{\Omega\omega}\right|^2$ $\rightarrow$ $\left\langle \hat{N}_\Omega \right\rangle = \left\langle 0_M \left| \hat{b}_\Omega^+ \hat{b}_\Omega^- \right| 0_M \right\rangle = \int_0^\infty d\omega\left|\beta_{\Omega\omega}\right|^2$ $\rightarrow$

$n_\Omega =$ $\left\langle \hat{N}_\Omega \right\rangle / V$ Bose-Einstein thermal distribution with defined temp

Additional Points to Consider (Not mentioned in M&W. Present author's perspective.)

The thermal bath of particles at the Unruh temperature (64) seen in the Rindler frame is *not* from the ½ quanta of the vacuum. It arises from the apparent difference between the Minkowski and Rindler frames, which gives rise to particles in the Rindler frame that are not seen in the Minkowski frame.

Indeed, if we include the ½ quanta terms (ignored before) in (8.43) [our (56)], then [see (6.43) pg 73, for number operator inside integral to find $\hat{H}$ including ½ quanta]

$$\left\langle \hat{N}_\Omega \right\rangle = \left\langle 0_M \left| \hat{b}_\Omega^+ \hat{b}_\Omega^- + \tfrac{1}{2}\delta^{(3)}(0) \right| 0_M \right\rangle \quad \text{LHS of M\&W (8.43) including ½ quanta contribution} \qquad (65)$$

The b operators, as we have seen in (8.43) to (8.46) give rise to the thermal bath seen in the Rindler frame. The ½ quanta terms are not involved. In fact, in terms of the number operator of (8.43) [our (65)] of the Rindler frame, the contribution is the same as that found in the Minkowski frame for $\hat{N}_\omega$, i.e., ½$\delta^{(3)}$(0).

In the next to last paragraph on pg 108, M&W seem to suggest the ½ quanta ("vacuum fluctuations") mediate the energy flow to the Rindler frame that causes the thermal bath. It is difficult for me to see how this works. There are no ½ quanta interaction terms in any of our relationships; no coupling between those terms and either the *a* or *b* particle types.

In this context, commonly heard claims that the Unruh effect provides evidence of the existence of the ½ quanta in the vacuum may seem perplexing.

Appendix A for these notes: The difference between accelerated metric in M&W and MTW

Compare M&W (8.28), pg 102, [our (18)] with unprimed accelerated frame coordinates to Misner, Thorne, and Wheeler's *Gravitation* (Freeman 1971) (MTW) (6.17) pg. 173, [our (66)] with primed accelerated frame coordinates. (Note MTW use a different sign convention for their metric. Their ds^2 = negative of M&W ds^2.)

$$dt^2 - dx^2 = \left(1 + a\xi^{1'}\right)^2 \left(d\xi^{0'}\right)^2 - \left(d\xi^{1'}\right)^2 \qquad \text{MTW } (6.18)\text{, pg. 173.} \qquad (66)$$

Where (8.27) [our (15)] compares to MTW (6.17), pg. 173 [our (67)]

$$t = \left(\frac{1}{a} + \xi^{1'}\right) \sinh a\xi^{0'} \qquad x = \left(\frac{1}{a} + \xi^{1'}\right) \cosh a\xi^{0'} \qquad \text{MTW (6.17), pg. 173} \qquad (67)$$

One set of coordinates can be transformed to the other with

$$\left(\frac{1}{a} + \xi^{1'}\right) = \frac{1}{a} e^{a\xi^1} \qquad \xi^{0'} = \xi^0 \qquad \rightarrow \qquad \xi^{1'} = \frac{1}{a}\left(e^{a\xi^1} - 1\right) \qquad \xi^{0'} = \xi^0 \qquad (68)$$

The M&W coordinates (unprimed) were chosen because M&W seek a conformally flat metric. (See (8.15), pg. 110, and comment above that equation.)

M&W Chap 8 Wholeness Chart for Unruh Radiation

Robert D. Klauber www.quantumfieldtheory.info

Key Step	**Math Relation**	**Note/Derivation**	**M&W**
Metric in inertial sys	$ds^2 = dt^2 - dx^2 \quad g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		(8.1)
Trajectory of accel observer (2D spacetime)	$\left.\begin{matrix} t = \frac{1}{a}\sinh a\tau \\ x = \frac{1}{a}\cosh a\tau \end{matrix}\right\} x^2 - t^2 = \frac{1}{a^2}$	Classical GR	(8.14) Fig. 8.1
Trajectory of each point in accel frame	$\left.\begin{matrix} t = \frac{e^{a\xi^1}}{a}\sinh a\xi^0 \\ x = \frac{e^{a\xi^1}}{a}\cosh a\xi^0 \end{matrix}\right\} x^2 - t^2 = \frac{e^{2a\xi^1}}{a^2} \qquad \xi^0 = \tau$	At origin (ξ^1 = 0), get (8.14) above	(8.27) Fig. 8.2
Metric in accel frame	$ds^2 = e^{2a\xi^1}\left[\left(d\xi^0\right)^2 - \left(d\xi^1\right)^2\right] \quad g_{\mu\nu} = \begin{bmatrix} e^{2a\xi^1} & 0 \\ 0 & -e^{2a\xi^1} \end{bmatrix} = e^{2a\xi^1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Find dt and dx from (8.27) & plug into (8.1)	(8.28)
Conformal transf result	$S = \frac{1}{2}\int\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right)dtdx = \frac{1}{2}\int\left(\left(\frac{\partial\phi}{\partial\xi^0}\right)^2 - \left(\frac{\partial\phi}{\partial\xi^1}\right)^2\right)d\xi^0 d\xi^1$	Due to conformal form of (8.28). Good for 2D.	(8.31)
Massless Scalar Field			
Inertial sys	$\hat{\phi} = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}}\frac{1}{\sqrt{2\omega}}\left[e^{-i\omega(t-x)}\hat{a}^-_\omega + e^{i\omega(t-x)}\hat{a}^+_\omega\right] + \left(\begin{matrix}\text{left-}\\ \text{moving}\end{matrix}\right)$	Sol to Eul-Lagr eq ($k = \omega$ for massless field)	(8.33)
Accel sys	$\hat{\phi} = \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}}\frac{1}{\sqrt{2\Omega}}\left[e^{-i\Omega(\xi^0-\xi^1)}\hat{b}^-_\Omega + e^{i\Omega(\xi^0-\xi^1)}\hat{b}^+_\Omega\right] + \left(\begin{matrix}\text{left-}\\ \text{moving}\end{matrix}\right)$	Same form as above due to (8.31)	"
Bogolyubov transf	$\hat{b}^-_\Omega = \int_0^\infty d\omega\left[\alpha_{\Omega\omega}\hat{a}^-_\omega + \beta_{\Omega\omega}\hat{a}^+_\omega\right]$	Complex conjug $= \hat{b}^+_\Omega$	(8.37)
Rel between $\alpha_{\Omega\omega}$ & $\beta_{\Omega\omega}$	$\left\|\alpha_{\Omega\omega}\right\|^2 = e^{\frac{2\pi\Omega}{a}}\left\|\beta_{\Omega\omega}\right\|^2$	(8.37) in (8.33) & manipulating	(8.42)
Occupatn num	$\left\langle \hat{N}_\Omega \right\rangle = \langle 0_M \| \hat{b}^+_\Omega \hat{b}^-_\Omega \| 0_M \rangle = \int d\omega \left\|\beta_{\Omega\omega}\right\|^2$	RHS from (8.37)	(8.43)
Normaliz cond	$\int_0^\infty d\omega\left[\left\|\alpha_{\Omega\omega}\right\|^2 - \left\|\beta_{\Omega\omega}\right\|^2\right] = \delta(0)$		(8.44)
	$\left\langle \hat{N}_\Omega \right\rangle = \frac{1}{\left(e^{\frac{2\pi\Omega}{a}} - 1\right)}\delta(0) \qquad \delta(0) = \frac{V_\infty}{(2\pi)^3}$	(8.42) in (8.44), $\int d\omega\left\|\beta_{\Omega\omega}\right\|^2$ in (8.43)	(8.45)
Num density	$n_\Omega = \frac{\left\langle \hat{N}_\Omega \right\rangle}{V_\infty} = \frac{1}{(2\pi)^3\left(e^{\frac{2\pi\Omega}{a}} - 1\right)}$	M&W omit factor of $(2\pi)^3$	(8.46)
Unruh temp	$T = \frac{a}{k2\pi}\ (\text{natural units}) \quad \rightarrow \quad T = \frac{a\hbar}{k2\pi c} = \left(1.60\times 10^{-17}\right)a\ (\text{MKS units})$	Compare (8.46) to Bose-Einstein distribution	(8.47)